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DATA ASSIMILATION FOR NON-LINEAR TIDAL MODELS

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SUMMARY

A data assimilation procedure to incorporate measurements into a non-linear tidal model by using Kalman-filtering techniques is developed. The Kalman filter is based on the two-dimensional shallow water equations. To account for the inaccuracies, these equations are embedded into a stochastic environment by introducing a coloured system noise process into the momentum equations. The continuity equation is assumed to be perfect. The deterministic part of the equations is discretized using an ADI method, the stochastic part using the Euler scheme. Assuming that the system noise is less spatially variable than the underlying water wave process, this stochastic part can be approximated on a coarser grid than the grid used to approximate the deterministic part. A Chandrasekhar-type filter algorithm is employed to obtain the constant-gain extended Kalman filter for weakly non-linear systems. The capabilities of the filter are illustrated by applying it to the assimilation of water level measurements into a tidal model of the North Sea.

KEY WORDS Data assimilation Kalman filtering Shallow water equations Water level measurements On-line prediction

1. INTRODUCTION

To describe and predict tidal flow, one usually employs deterministic numerical models based on the non-linear shallow water equations. When the meteorological effects are small or can be neglected, these models produce accurate results. However, during stormy periods, deterministic models are far from perfect. Errors are introduced by fluctuations in the meteorological input or by the poorly known influence of the wind on the water movement described by the wind friction coefficient. Furthermore, considerable uncertainty is associated with the open boundary conditions, since it is possible that severe storms outside the domain of the problem create surges which propagate across such a boundary into the model. Therefore a number of statistical methods, i.e. simple empirical or black box models derived from series of observations, have been developed for the on-line prediction of storm surges. These models have the capability of incorporating on-line measurements into the prediction process to adapt the predictions continuously to changing conditions. However, since the deterministic numerical models have a more physical basis and provide a more realistic description of the water movement, the use of a data assimilation technique to incorporate the measurements into a numerical model is of course preferable.

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Data assimilation techniques have received considerable attention in the meteorological literature.[1] The most common data assimilation technique used in numerical weather prediction is optimal interpolation, in which some estimates of the error statistics of the numerical model are used to correct the results of the model using the measurements available. However, since these error statistics have to be determined by adopting some *ad hoc* statistical assumptions, the correction produced by optimal interpolation is not consistent with the underlying numerical model. As a consequence, in the case of a shallow water flow model describing the complicated flow pattern due to an irregular geometry, the use of optimal interpolation yields unrealistic corrections or introduces instabilities.

A new approach to data assimilation is Kalman filtering. In order to use a Kalman filter for incorporating data into a numerical shallow water flow model, the model is embedded into a stochastic environment by introducing a system noise process. In this way it is possible to take into account the inaccuracies of the underlying deterministic system. By using a Kalman filter, the information provided by the resulting stochastic–dynamic model and the noisy measurements taken from the actual system are combined to obtain an optimal (least-squares) estimate of the state of the system. Kalman filtering is similar to optimal interpolation. However, with a Kalman filter the error statistics of the numerical model are determined by using the stochastic extension of the model. Therefore the correction produced by this filter, unlike in the case of optimal interpolation, is guaranteed to be consistent with this stochastic model, even in the case of very irregular flow patterns.

In the last decade, Kalman filtering has gained acceptance as a powerful framework for data assimilation, e.g. in meteorology,[2] oceanography[3] and hydraulics.[4,5] However, most applications of Kalman-filtering techniques deal with one-dimensional models. Although the extension of these one-dimensional filtering techniques to two space dimensions would not give rise to conceptual problems, it would impose an unacceptably greater computational burden. In order to obtain a computationally efficient filter, simplifications have to be introduced. Unfortunately, there are serious problems with the more obvious simplifications one might consider.[3,6] As a consequence, Kalman-filtering techniques have to date seldom been applied to realistic two-dimensional filtering problems. Parrish and Cohn[7] developed a filter for a linear two-dimensional numerical model based on the assumption that errors at large distance points are not correlated. As a result the computational complexity of the filter can be reduced dramatically, and they showed that their filter is indeed computationally feasible for numerical weather prediction. However, for most shallow water flow problems the domain of the problem is relatively small, so that errors are highly correlated in space and this approach cannot be employed. Therefore Heemink[6,8] developed a time-invariant filter approach based on the linear two-dimensional shallow water equations. By using a Chandrasekhar-type filter algorithm, the special structure of the filtering problem is exploited to obtain an efficient implementation.

All previous applications of filtering theory to two-dimensional shallow water flow identification problems have been based on a linear deterministic model. In this paper, building on the ideas presented in Reference 6, we develop a data assimilation procedure for weakly non-linear tidal models. The approach is based on the non-linear two-dimensional shallow water equations. In Section 2 these equations as well as the boundary conditions are embedded into a stochastic environment. The equations are approximated numerically in Section 3 by using an ADI scheme for the deterministic part and the Euler scheme for the stochastic part of the equations. By defining a state vector that consists of the water levels and velocities at all the grid points as well as the uncertain parameters introduced into the boundary conditions, the model is rewritten in state-space form in Section 4. Employing a constant-gain extended Kalman filter for weakly non-linear systems in Section 5, the on-line measurements of the water level available can be used to estimate the shallow water flow. By using this suboptimal filter approach, these estimates are determined using the non-linear shallow water equations, while the time-consuming second-moment equations are solved using the linearized ones. Since the filter is time-invariant, these second-moment computations do not have to be recomputed as new measurements become available, but can be solved once and off-line. Furthermore, the fact that the system noise is less spatially variable than the underlying process can be exploited to reduce the computational burden by employing a Chandrasekhar-type algorithm. In Section 6 a number of applications of the approach are described in detail to illustrate the performance of the filter.

2. STOCHASTIC SHALLOW WATER EQUATIONS

The two-dimensional non-linear shallow water equations do not perfectly describe the shallow water flow in coastal waters. Therefore we embed these equations into a stochastic environment. The resulting momentum equations are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial h}{\partial x}-fv+\lambda(u,v)\frac{u}{D+h}=\tau_x+S_u-\frac{1}{\rho_w}\frac{\partial p_a}{\partial x}, \tag{1}$$

$$\mathrm{d}S_u=-\alpha_1 S_u\,\mathrm{d}t+\sigma\,\mathrm{d}W_u, \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial h}{\partial y}+fu+\lambda(u,v)\frac{v}{D+h}=\tau_y+S_v-\frac{1}{\rho_w}\frac{\partial p_a}{\partial y}, \tag{3}$$

$$\mathrm{d}S_v=-\alpha_1 S_v\,\mathrm{d}t+\sigma\,\mathrm{d}W_v, \tag{4}$$

where h is the water level, u and v are the water velocities in the x- and y-directions respectively, D is the depth of water, f is the Coriolis parameter, λ is the bottom friction coefficient, τ_x and τ_y (equal to $\gamma V^2\cos\psi$ and $\gamma V^2\sin\psi$ respectively, with γ the wind friction coefficient, V the wind speed and ψ the wind direction) are the wind stresses in the x- and y-directions respectively, ρ_w is the density of water, p_a is atmospheric pressure, g is the acceleration due to gravity, S_u and S_v are coloured system noise processes, W_u and W_v are Brownian motion processes with statistics

$$\mathrm{E}\{\mathrm{d}W_u(x,y,t)\}=0, \tag{5}$$

$$\mathrm{E}\{\mathrm{d}W_u(x_1,y_1,t)\,\mathrm{d}W_u(x_2,y_2,t)\}=\exp\{-\delta\sqrt{[(x_1-x_2)^2+(y_1-y_2)^2]}\}\,\mathrm{d}t, \tag{6}$$

$$\mathrm{E}\{\mathrm{d}W_v(x,y,t)\}=0, \tag{7}$$

$$\mathrm{E}\{\mathrm{d}W_v(x_1,y_1,t)\,\mathrm{d}W_v(x_2,y_2,t)\}=\exp\{-\delta\sqrt{[(x_1-x_2)^2+(y_1-y_2)^2]}\}\,\mathrm{d}t, \tag{8}$$

$$\mathrm{E}\{\mathrm{d}W_u(x_1,y_1,t)\,\mathrm{d}W_v(x_2,y_2,t)\}=0, \tag{9}$$

and α_1, σ and δ are constants.

Here we note that since σ is constant, the stochastic differential equations (2) and (4) are the same in the Îto and Stratonowitz sense and can be manipulated by formal rules.[9]

By introducing the system noise processes into the momentum equations according to equations (1)–(9), it is assumed that the largest errors of the underlying deterministic model are introduced by the uncertainty in the wind stress input, i.e. by the fluctuations in the wind input or by the poorly known influence of the wind on the water movement described by the wind friction coefficient. In this way the coloured noise processes S_u and S_v are corrections on the wind stress inputs τ_x and τ_y respectively. The statistics of S_u and S_v can be determined easily:[9]

$$\mathrm{E}\{S_u(x,y,t)\}=0, \tag{10}$$

$$E\{S_u(x_1, y_1, t_1) S_u(x_2, y_2, t_2)\} = (\sigma^2 \exp\{-\alpha_1(t_2 - t_1) - \delta\sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}\})/2\alpha_1, \tag{11}$$

$$E\{S_v(x, y, t)\} = 0, \tag{12}$$

$$E\{S_v(x_1, y_1, t_1) S_v(x_2, y_2, t_2)\} = (\sigma^2 \exp\{-\alpha_1(t_2 - t_1) - \delta\sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}\})/2\alpha_1, \tag{13}$$

$$E\{S_u(x_1, y_1, t_1) S_v(x_2, y_2, t_2)\} = 0. \tag{14}$$

The parameters α_1 and δ are measures of respectively the temporal and spatial variability of the errors concerned with the momentum equations.

While the noise processes W_u and W_v are introduced to model the uncertainty associated with the momentum equations, the continuity equation is assumed to be perfect:

$$\frac{\partial h}{\partial t} + \frac{\partial[u(D+h)]}{\partial x} + \frac{\partial[v(D+h)]}{\partial y} = 0. \tag{15}$$

When the water is rather deep, the non-linearities in the equations (1), (3) and (15) are small and it is possible to approximate these equations by the linear ones:

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} - fv + \lambda\frac{u}{D} = \tau_x + S_u - \frac{1}{\rho_w}\frac{\partial p_a}{\partial x}, \tag{16}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial h}{\partial y} + fu + \lambda\frac{v}{D} = \tau_y + S_v - \frac{1}{\rho_w}\frac{\partial p_a}{\partial y}, \tag{17}$$

$$\frac{\partial h}{\partial t} + \frac{\partial(uD)}{\partial x} + \frac{\partial(vD)}{\partial y} = 0. \tag{18}$$

The wave motion is completely described by the non-linear equations (1)–(9) and (15) or the linear ones (16), (2), (17), (4)–(9) and (18) provided that initial values and closed and open boundary conditions are given. At a closed boundary the velocity normal to the boundary is zero:

$$v_\perp = 0. \tag{19}$$

There is usually a considerable uncertainty associated with the open boundary conditions, since it is possible that severe storms outside the domain of the problem create surges which propagate across a boundary into the model. To model these uncertainties, a stochastic water level boundary condition is applied at an open boundary:

$$h = f(t) + H_b, \tag{20}$$

$$dH_b = -\alpha_2 H_b\, dt + \sigma_b\, dW_b, \tag{21}$$

Here α_2 and σ_b are constants. The Brownian motion process W_b is assumed to be independent of the other system noise components, with statistics

$$E\{dW_b(t)\} = 0, \tag{22}$$

$$E\{dW_b(t)\, dW_b(t)\} = \sigma_b^2 dt. \tag{23}$$

As a result the statistics of the coloured noise process H_b are

$$E\{H_b(t)\} = 0, \tag{24}$$

$$E\{H_b(t_1)H_b(t_2)\} = \{\sigma_b^2 \exp[-\alpha_2(t_2 - t_1)]\}/2\alpha_2 \tag{25}$$

In the non-linear case it has also to be assumed that at the inflow the velocity parallel to the open boundary is zero:

$$v_{\parallel}=0. \tag{26}$$

All the parameters of the noise processes have to be specified. Unfortunately this knowledge is very poor. However, as is well known, the filter estimates are not very sensitive to the noise statistics.[10,11]

3. NUMERICAL APPROXIMATION

The deterministic part of the shallow water equations is approximated by using an ADI method. The numerical scheme was developed by Stelling[12] and is based on the well known work of Leendertse.[13] It is unconditionally stable and lacks artificial viscosity. Defining a space-staggered grid G_1, the finite difference equations to approximate the homogeneous part of the linear equations are as follows.

Step 1

explicit

$$v^{k+1/2}_{m,n+1/2}=\left(v^k_{m,n+1/2}-g\frac{\Delta t}{2\Delta y}(h^k_{m,n+1}-h^k_{m,n})+\tfrac{1}{8}\Delta t f(u^k_{m-1/2,n+1}+u^k_{m-1/2,n}\right.$$
$$\left.+u^k_{m+1/2,n+1}+u^k_{m+1/2,n})\right)\Big/\left(1+\frac{\Delta t\lambda}{D_{m-1/2,n+1/2}+D_{m+1/2,n+1/2}}\right), \tag{27}$$

implicit

$$(h^{k+1/2}_{m,n}-h^k_{m,n})+\frac{\Delta t}{4\Delta x}[(D_{m+1/2,n+1/2}+D_{m+1/2,n-1/2})u^{k+1/2}_{m+1/2,n}$$
$$-(D_{m-1/2,n+1/2}+D_{m-1/2,n-1/2})u^{k+1/2}_{m-1/2,n}]+\frac{\Delta t}{4\Delta y}[(D_{m+1/2,n+1/2}$$
$$+D_{m-1/2,n+1/2})v^k_{m,n+1/2}-(D_{m+1/2,n-1/2}+D_{m-1/2,n-1/2})v^k_{m,n-1/2}]=0, \tag{28}$$

$$(u^{k+1/2}_{m+1/2,n}-u^k_{m+1/2,n})+g\frac{\Delta t}{2\Delta x}(h^k_{m+1,n}-h^k_{m,n})-\tfrac{1}{8}\Delta t f(v^{k+1/2}_{m,n+1/2}+v^{k+1/2}_{m+1,n+1/2}$$
$$+v^{k+1/2}_{m+1,n-1/2}+v^{k+1/2}_{m,n-1/2})+\frac{\lambda\Delta t u^{k+1/2}_{m+1/2,n}}{D_{m+1/2,n+1/2}+D_{m+1/2,n-1/2}}=0. \tag{29}$$

Step 2

explicit

$$u^{k+1}_{m+1/2,n}=\left(u^{k+1/2}_{m+1/2,n}-g\frac{\Delta t}{2\Delta x}(h^{k+1/2}_{m+1,n}-h^{k+1/2}_{m,n})-\tfrac{1}{8}\Delta t f(v^{k+1/2}_{m+1,n-1/2}+v^{k+1/2}_{m,n-1/2}\right.$$
$$\left.+v^{k+1/2}_{m+1,n+1/2}+v^{k+1/2}_{m,n+1/2})\right)\Big/\left(1+\frac{\Delta t\lambda}{D_{m+1/2,n+1/2}+D_{m+1/2,n-1/2}}\right), \tag{30}$$

implicit

$$(h_{m,n}^{k+1} - h_{m,n}^{k+1/2}) + \frac{\Delta t}{4\Delta x}[(D_{m+1/2,n+1/2} + D_{m+1/2,n-1/2})u_{m+1/2,n}^{k+1/2}$$

$$- (D_{m-1/2,n+1/2} + D_{m-1/2,n-1/2})u_{m-1/2,n}^{k+1/2}] + \frac{\Delta t}{4\Delta y}[(D_{m+1/2,n+1/2}$$

$$+ D_{m-1/2,n+1/2})v_{m,n+1/2}^{k+1} - (D_{m+1/2,n-1/2} + D_{m-1/2,n-1/2})v_{m,n-1/2}^{k+1}] = 0, \tag{31}$$

$$(v_{m,n+1/2}^{k+1} - v_{m,n+1/2}^{k+1/2}) + g\frac{\Delta t}{2\Delta y}(h_{m,n+1}^{k+1} - h_{m,n}^{k+1}) + \tfrac{1}{8}\Delta t f(u_{m-1/2,n}^{k+1} + u_{m-1/2,n+1}^{k+1}$$

$$+ u_{m+1/2,n+1}^{k+1} + u_{m+1/2,n}^{k+1}) + \frac{\lambda \Delta t v_{m,n+1/2}^{k+1}}{D_{m-1/2,n+1/2} + D_{m+1/2,n+1/2}} = 0. \tag{32}$$

For details such as the numerical boundary treatment we refer to Reference 12.

The stochastic differential equations are approximated using the Euler scheme:

$$S_u^{k+1} = (1 - \alpha_1 \Delta t)S_u^k + \sigma(W_u^{k+1} - W_u^k), \tag{33}$$

$$S_v^{k+1} = (1 - \alpha_1 \Delta t)S_v^k + \sigma(W_v^{k+1} - W_v^k). \tag{34}$$

The meteorological input processes τ_x, τ_y and p_a as well as the noise processes S_u and S_v are defined on a grid G_2. Since these processes are (assumed to be) less spatially variable than the underlying water wave process, G_2 is much coarser than G_1 and therefore interpolation is necessary to obtain these processes in the grid points of G_1.

The numerical treatment of the stochastic open boundary condition is similar to the approximation of the system noise processes.

4. STATE-SPACE REPRESENTATION OF THE MODEL

The state-space representation of the linear model is obtained by defining an n-vector $\mathbf{X}_{t_k}$ that contains the water level and velocities at all the grid points of G_1. The finite difference scheme can now be rewritten as a deterministic system

$$\mathbf{A}\mathbf{X}_{t_{k+1/2}} = \mathbf{B}\mathbf{X}_{t_k} + \Lambda u_{t_{k+1/2}}, \tag{35}$$

$$\mathbf{C}\mathbf{X}_{t_{k+1}} = \mathbf{D}\mathbf{X}_{t_{k+1/2}} + \Lambda u_{t_{k+1}}. \tag{36}$$

Here $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are coefficient matrices representing the ADI scheme and u_{t_k} is the meteorological input given on the grid G_2. The matrix Λ represents a sequence of linear operations to interpolate the meteorological input at the grid points of G_1. Similarly, we define the q-vector $\mathbf{P}_{t_k}$ to contain the coloured noise processes introduced in the open boundary conditions and the p-vector $\mathbf{S}_{t_k}$ to consist of the system noise process at all the grid points of G_2. Since the system noise is less spatially variable than the water wave process, we have $p \ll n$. Introducing the noise process only in the second half time step of the finite difference equations, the stochastic–dynamic model can be described by

$$\begin{bmatrix} \mathbf{C} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t_{k+1}} \\ \mathbf{P}_{t_{k+1}} \\ \mathbf{S}_{t_{k+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{DA}^{-1}\mathbf{B} & \mathbf{E} & \Lambda \\ 0 & \mathbf{I}-\alpha_2 \Delta t \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I}-\alpha_1 \Delta t \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{t_k} \\ \mathbf{P}_{t_k} \\ \mathbf{S}_{t_k} \end{bmatrix} + \begin{bmatrix} \mathbf{DA}^{-1}\Lambda \\ 0 \\ 0 \end{bmatrix} \mathbf{u}_{t_{k+1/2}}$$

$$+ \begin{bmatrix} \Lambda \\ 0 \\ 0 \end{bmatrix} \mathbf{u}_{t_{k+1}} + \begin{bmatrix} 0 \\ \mathbf{W}^{\mathrm{b}}_{t_{k+1}} \\ \mathbf{W}_{t_{k+1}} \end{bmatrix}. \tag{37}$$

Here $\mathbf{E}$ is a coefficient matrix representing the open boundary treatment and $\mathbf{W}^{\mathrm{b}}_{t_k}$ contains the increments of the Brownian motion processes introduced in the open boundary conditions. Similarly, $\mathbf{W}_{t_k}$ is a p-vector consisting of the noise components at all the grid points of G_2. The covariance matrices of $\mathbf{W}^{\mathrm{b}}_{t_k}$ and $\mathbf{W}_{t_k}$ can be derived easily from equations (22)–(23) and (5)–(9) respectively. The model is of the hyperbolic type. As a consequence the effect of the initial condition, which is often poorly known, vanishes after a limited period of time. Equation (37) can be rewritten as

$$\mathbf{A}^*\mathbf{Y}_{t_{k+1}} = \mathbf{B}^*\mathbf{Y}_{t_k} + \mathbf{u}^*_{t_{k+1}} + \mathbf{W}^*_{t_{k+1}}, \tag{38}$$

and Q^* as the covariance matrix of $\mathbf{W}^*_t$.

In the non-linear case the stochastic–dynamic system representation of the model becomes

$$\mathbf{A}^*\mathbf{Y}_{t_{k+1}} = \mathbf{b}^*(\mathbf{Y}_{t_k}, \mathbf{u}^*_{t_{k+1}}) + \mathbf{W}^*_{t_{k+1}}. \tag{39}$$

5. KALMAN FILTERING

Assuming that measurements of the water level are available at m grid points of G_1, the observation equation can be derived easily:

$$\mathbf{Z}_{t_k} = \mathbf{M}\mathbf{Y}_{t_k} + \mathbf{V}_{t_k}, \tag{40}$$

where the m-vector $\mathbf{Z}_{t_k}$ contains the measurements taken at time $k\Delta t$ and $\mathbf{V}_{t_k}$ is the white measurement noise with covariance matrix $\mathbf{R}$. $\mathbf{V}_{t_k}$ and $\mathbf{W}^*_{t_k}$ are assumed to be mutually independent. In practice we may assume that the measurement errors at different locations are mutually independent and have equal variance, so $\mathbf{R}$ is diagonal with elements r^2. If necessary, equation (40) can be modified to account for the fact that not all measurements are available at grid points.

It is desired to combine the measurements taken from the actual system and modelled by relation (40) with the information provided by the system model (38) or (39) to obtain an estimate of the system state $\mathbf{Y}_{t_k}$. Let us first consider the linear case. If $\hat{\mathbf{Y}}(k|l)$ is defined as the least-squares estimate of the state $\mathbf{Y}_{t_k}$ using the measurements $\mathbf{Z}_{t_1}, \mathbf{Z}_{t_2}, \ldots, \mathbf{Z}_{t_l}$, $l \leqslant k$, the recursive filter equations to obtain this quantity can be summarized as follows:[6]

$$\mathbf{A}^*\hat{\mathbf{Y}}(k+1|k) = \mathbf{B}^*\hat{\mathbf{Y}}(k|k) + u^*_{t_{k+1}}, \tag{41}$$

$$\hat{\mathbf{Y}}(k+1|k+1) = \hat{\mathbf{Y}}(k+1|k) + \mathbf{K}[\mathbf{Z}_{t_{k+1}} - \mathbf{M}\hat{\mathbf{Y}}(k+1|k)], \tag{42}$$

where the steady state Kalman filter gain $\mathbf{K}$ can be determined by the Chandrasekhar-type equations[14]

$$\mathbf{A}^*\mathbf{Y}(k+1) = \mathbf{B}^*\mathbf{S}(k), \tag{43}$$

$$\mathbf{G}(k+1) = \mathbf{G}(k) + \mathbf{Y}(k+1)\mathbf{L}(k)\mathbf{Y}(k+1)^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}, \tag{44}$$

$$\mathbf{R}^{\varepsilon}(k+1) = \mathbf{R}^{\varepsilon}(k) + \mathbf{M}\mathbf{Y}(k+1)\mathbf{L}(k)\mathbf{Y}(k+1)\mathbf{M}^{\mathrm{T}}, \tag{45}$$

$$\mathbf{K}(k+1) = \mathbf{G}(k+1)\mathbf{R}^{\varepsilon}(k+1)^{-1}, \tag{46}$$

$$\mathbf{S}(k+1) = (\mathbf{I} - \mathbf{K}\mathbf{M})\mathbf{Y}(k+1), \tag{47}$$

$$\mathbf{L}(k+1) = \mathbf{L}(k) + \mathbf{L}(k)\mathbf{Y}(k+1)^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{R}^{\varepsilon}(k)^{-1}\mathbf{M}\mathbf{Y}(k+1)\mathbf{L}(k), \tag{48}$$

with initial conditions

$$\mathbf{A}^*\mathbf{Y}(1) = \mathbf{\Lambda}, \tag{49}$$

$$\mathbf{G}(0) = 0, \tag{50}$$

$$\mathbf{R}^{\varepsilon}(0) = \mathbf{R}, \tag{51}$$

$$\mathbf{L}(0) = \mathbf{Q}^*. \tag{52}$$

The equations are iterated until

$$\|\mathbf{K}(k+1) - \mathbf{K}(k)\| < \varepsilon \|\mathbf{K}(k)\|, \tag{53}$$

where ε is prespecified. Since the underlying deterministic system is of the hyperbolic type, the number of iterations required depends on the travelling times of the waves in the model and therefore on the size of the domain of the problem.

Let us now consider the non-linear model (39). The linearized Kalman filter can in this case be summarized as follows:

$$\mathbf{A}^*\hat{\mathbf{Y}}(k+1|k) = \mathbf{b}^*(\hat{\mathbf{Y}}(k|k)) + \mathbf{u}^*_{t_{k+1}}, \tag{54}$$

$$\hat{\mathbf{Y}}(k+1|k+1) = \hat{\mathbf{Y}}(k+1|k) + \mathbf{K}_{t_k}[\mathbf{Z}_{t_{k+1}} - \mathbf{M}\hat{\mathbf{Y}}(k+1|k)]. \tag{55}$$

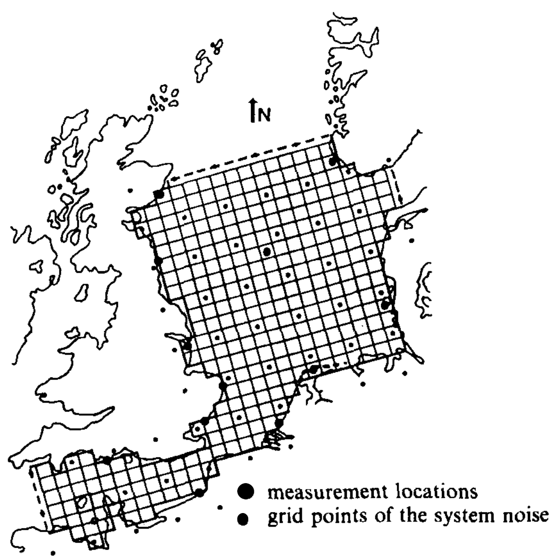

Figure 1. North Sea model

The time-varying Kalman filter gain $\mathbf{K}_{t_k}$ is determined by linearizing $\mathbf{b}^*$ about a reference trajectory and by using the filter equations for linear time-varying systems.[9] In our case the time variations of $\mathbf{K}_{t_k}$ are only caused by the non-linearities of $\mathbf{b}^*$. However, if the water is rather deep, these non-linearities are relatively small and the time variations of $\mathbf{K}_{t_k}$ are small too. Moreover, since deterministic tidal models usually provide a reasonable description of the tidal movement, the correction produced by the Kalman filter using the measurements is small in relation to the deterministic results. Therefore a relatively small error in $\mathbf{K}_{t_k}$ hardly affects the estimates of the filter and we may expect that a constant-gain extended Kalman filter will produce estimates that are nearly optimal.[15] This filter is based on the linearization of $\mathbf{b}^*$ about an equilibrium state and, as a consequence, in our case this filter is time-invariant:

$$\mathbf{A}^*\hat{\mathbf{Y}}(k+1\,|\,k) = \mathbf{b}^*(\hat{\mathbf{Y}}(k\,|\,k)) + \mathbf{u}^*_{t_{k+1}}, \tag{56}$$

$$\hat{\mathbf{Y}}(k+1\,|\,k+1) = \hat{\mathbf{Y}}(k+1\,|\,k) + \mathbf{K}[\mathbf{Z}_{t_{k+1}} - \mathbf{M}\hat{\mathbf{Y}}(k+1\,|\,k)], \tag{57}$$

where the Kalman gain is determined by solving equations (43)–(52) based on the linear model (38).

6. FILTERING RESULTS

Before the filter can be safely applied to prototype situations, it is necessary to demonstrate that it can perform adequately under known conditions. Therefore we first developed a filter based on a relatively simple numerical model describing the water movement in the North Sea and the English Channel. In Figure 1 the grid G_1 covering this area is shown. Here $\Delta x = 18{\cdot}5$ km, $\Delta y = 19$ km and $\Delta t = 10$ min. The grid G_2 and the measurement locations are also shown in Figure 1. The parameters of the noise processes are chosen to be $\sigma = 10^{-4}$, $\alpha_1 = 0{\cdot}5 \times 10^{-4}$, $\delta = 10^{-6}$, $\sigma_b = 0{\cdot}5 \times 10^{-4}$ and $\alpha_2 = 0{\cdot}5 \times 10^{-4}$. Water level data are created by using the model and a realistic meteorological input during a stormy period. At the open boundary we do not consider the tide and prescribe the water level to be zero. Measurement errors with $r = 5$ cm are simulated by means of a random generator. This data set was used to estimate the water levels and velocities. While filtering the data it was assumed that the meteorological input was zero.

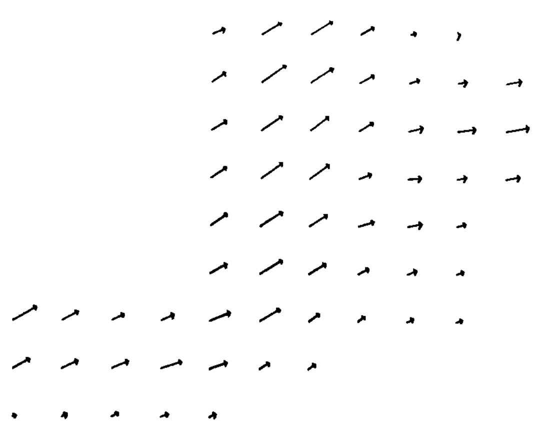

Figure 2. Wind velocities

Therefore without using the generated data the estimates of the water velocities produced by the filter would be zero too. Of course, this experiment describes an extreme situation in which the filter has to reconstruct the entire storm. As a representative example, in Figures 2–4 respectively the wind velocities, the simulated water velocities and the filtered water velocities at a certain time are shown. These and similar results show that, considering the limited number of measurement locations available, the filter performance is excellent.[8]

Numerous experiments using field data have been performed with a model that covers the entire Continental Shelf as shown in Figures 5 and 6.[16] The on-line measurement stations that are available in the North Sea are also shown in Figure 5. To illustrate the correction of the water velocities produced by the filter, in Figures 7 and 8 respectively the deterministic and filtered flow patterns in the southern North Sea at a certain time are drawn. Of course the correction is small

Figure 3. Simulated water velocities

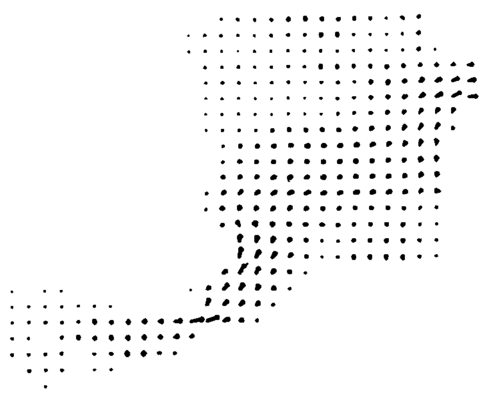

Figure 4. Filtered water velocities

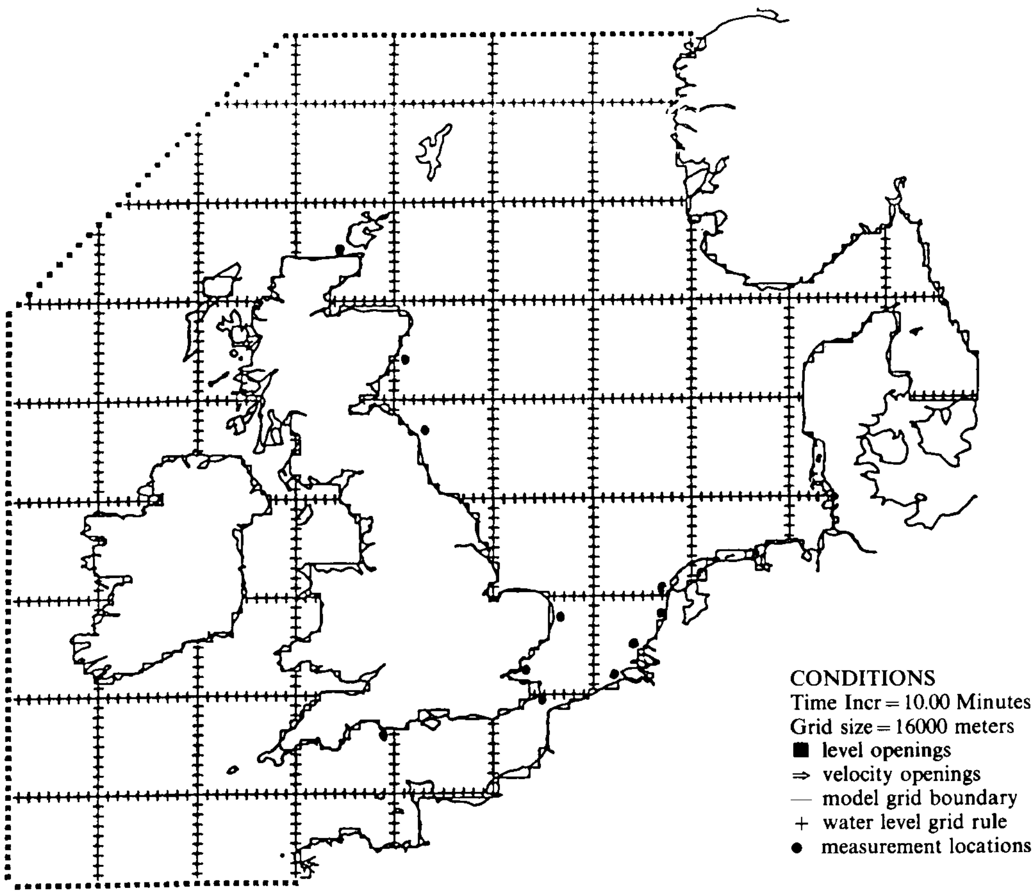

Figure 5. Continental Shelf model

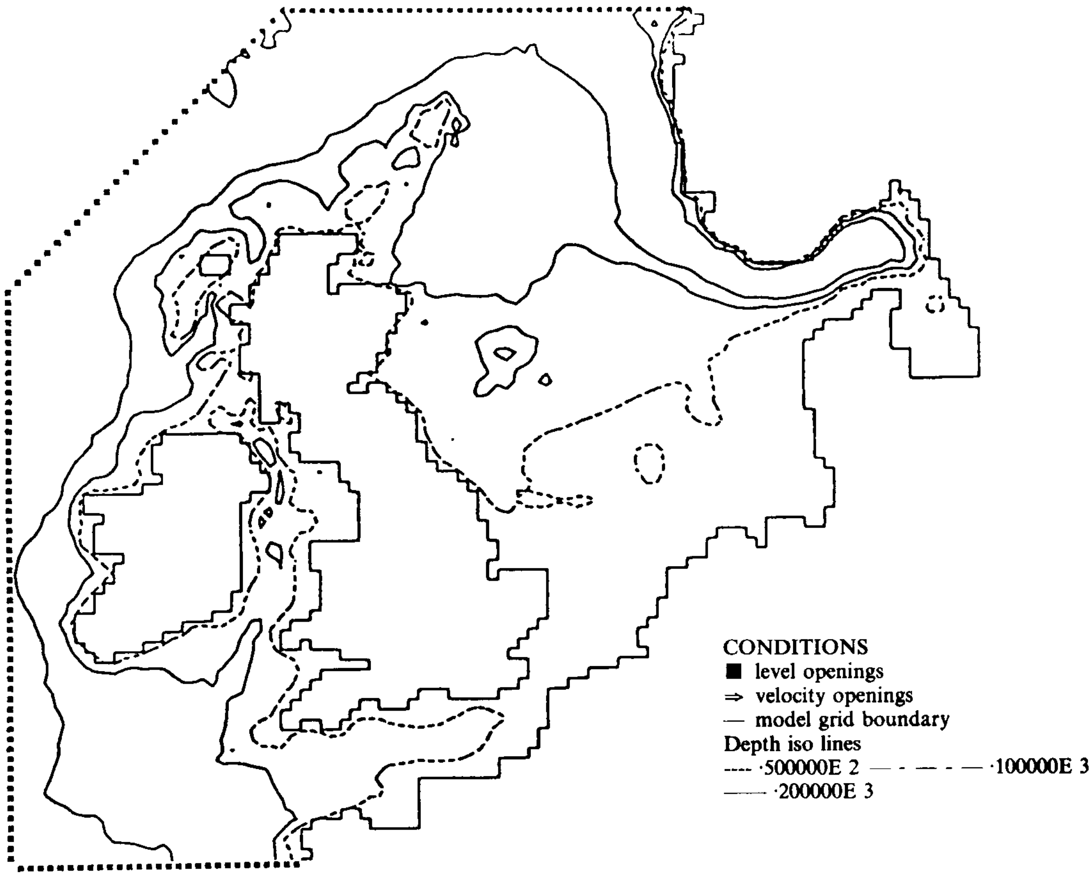

Figure 6. Depth contours of the Continental Shelf model

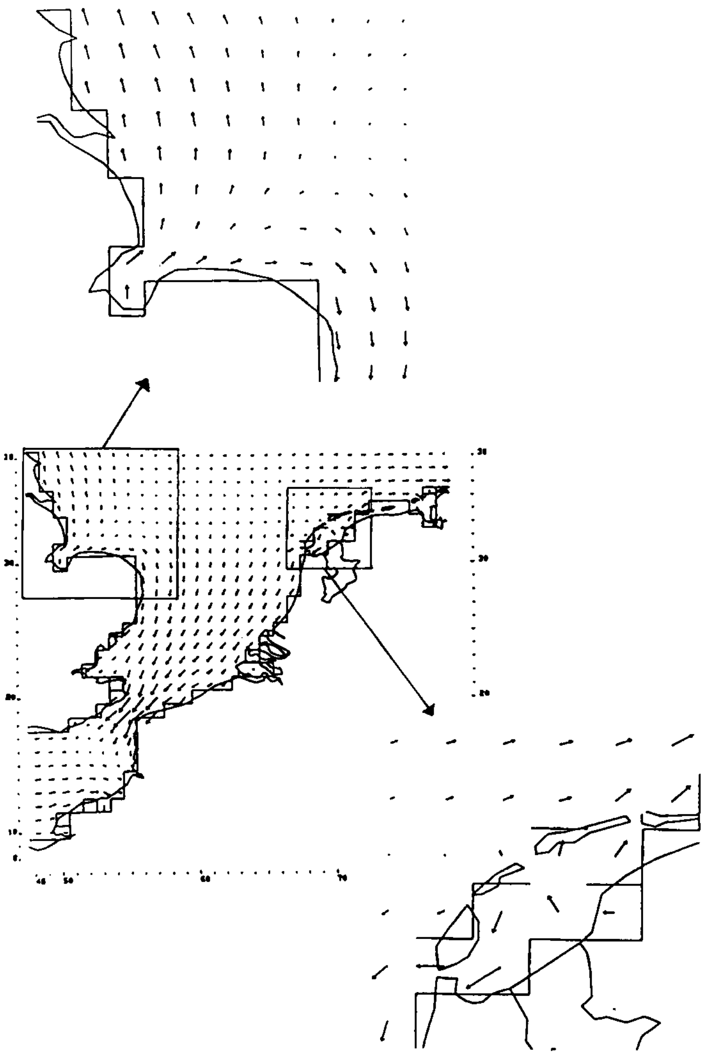

Figure 7. Deterministic flow pattern

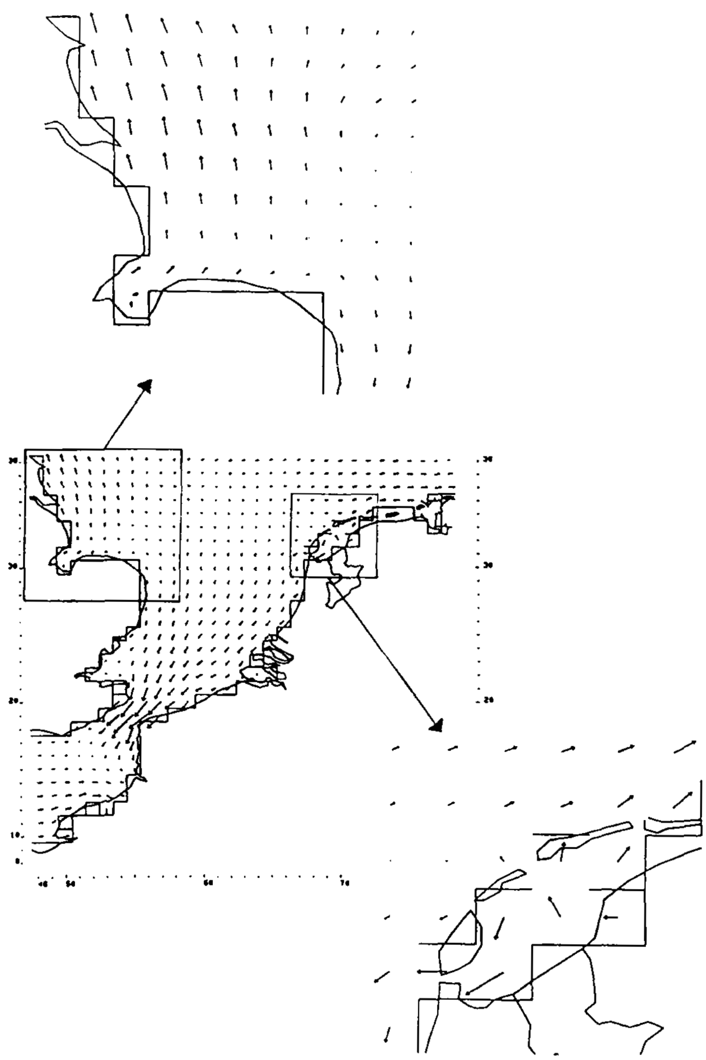

Figure 8. Filtered flow pattern

with respect to the tidal flow. Clearly the filter does not introduce instabilities. To demonstrate the improvement of the water level predictions during stormy conditions obtained by using the water level data, in Figures 9 and 10 the deterministic as well as the filter predictions at respectively Southend and Ijmuiden are shown. Here the filter used the data up to 14 February, 2:00 h. Clearly the short-term predictions of the filter are much more accurate. Of course the improvement obtained by filtering the data available decreases with the prediction interval.

7. CONCLUSIONS

In this paper we have developed a data assimilation procedure based on Kalman filtering to incorporate water level measurements into a non-linear numerical tidal model. Numerous

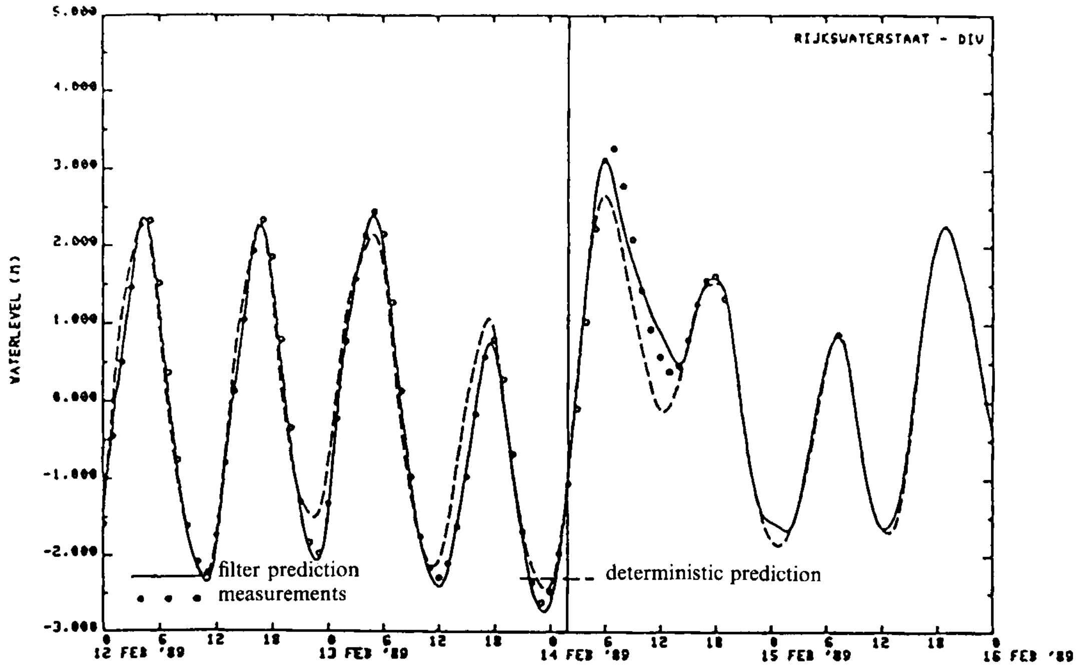

Figure 9. Prediction of the water level at Southend

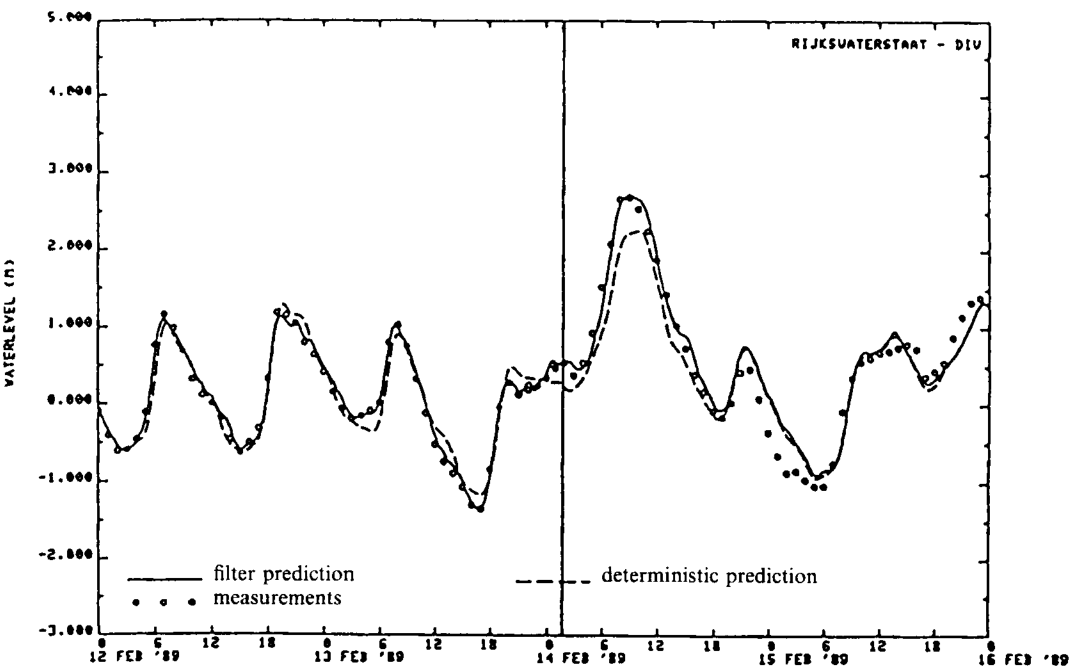

Figure 10. Prediction of the water level at Ijmuiden

experiments with simulated as well as field data show excellent filter performance and indicate that, in the weakly non-linear case, the suboptimal filter is very robust and does not introduce instabilities. The approach has recently been implemented to predict the water level along the Dutch coast on a routine basis.

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